ON THE NUMBER OF INDECOMPOSABLE TOTALLY REFLEXIVE MODULES

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Abstract. In this note, it is proved that over a commutative noetherian henselian non-Gorenstein local ring there are infinitely many isomorphism classes of indecomposable totally reflexive modules, if there is a nonfree cyclic totally reflexive module.

1. Introduction

Throughout this note, we assume that all rings are commutative and noetherian, and that all modules are finitely generated.

In the late 1960s, Auslander [1] introduced a homological invariant for modules which is called Gorenstein dimension, or G-dimension for short. After that, he further developed the theory of G-dimension with Bridger [2]. Many properties enjoyed by G-dimension are analogous to those of projective dimension. An important feature is that G-dimension characterizes Gorenstein local rings exactly as projective dimension characterizes regular local rings. A module of G-dimension zero is called a totally reflexive module. Avramov and Martsinkovsky [3] and Holm [7] proved that over a local ring any module M of finite G-dimension admits an exact sequence $0 \to Y \to X \to M \to 0$ such that X is totally reflexive and Y is of finite projective dimension. This result says that in the study of modules of finite G-dimension it is essential to consider totally reflexive modules.

On the other hand, Cohen-Macaulay local rings of finite Cohen-Macaulay type, namely Cohen-Macaulay local rings over which there are only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules, have been deeply studied since the work of Herzog [6]. Under a few assumptions, Gorenstein local rings of finite Cohen-Macaulay type are hypersurfaces, and they have been classified completely. Moreover, all isomorphism classes of indecomposable maximal Cohen-Macaulay modules over them are described concretely; see [14] for the details.

Over a Gorenstein local ring, totally reflexive modules are the same as maximal Cohen-Macaulay modules. Hence it is natural to expect that totally reflexive modules over an arbitrary local ring may behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring, and we are interested in local rings over which there are only finitely many isomorphism classes of indecomposable totally reflexive modules; we want to determine all such isomorphism classes. However, we guess that such ring cannot essentially exist in the non-Gorenstein case:

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Conjecture 1.1. Let R be a non-Gorenstein local ring. Suppose that there is a nonfree totally reflexive R-module. Then there are infinitely many isomorphism classes of indecomposable totally reflexive R-modules.

In this conjecture, so as to exclude the case where all totally reflexive modules are free, it is assumed that there is a nonfree totally reflexive module. Indeed, for instance, over a Cohen-Macaulay non-Gorenstein local ring with minimal multiplicity, every totally reflexive module is free [15].

The author proved that the above conjecture is true over a henselian local ring of low depth:

Theorem 1.2. [11][12][13] *Let R be a henselian non-Gorenstein local ring of depth at most two. Suppose that there is a nonfree totally reflexive R-module. Then there are infinitely many isomorphism classes of indecomposable totally reflexive R-modules.*

The main purpose of this note is to prove that the conjecture is true over a henselian local ring having a nonfree cyclic totally reflexive module.

Theorem 1.3. *Let R be a henselian non-Gorenstein local ring. Suppose that there is a nonfree cyclic totally reflexive R-module. Then there are infinitely many isomorphism classes of indecomposable totally reflexive R-modules.*

This theorem says, for example, that if R is a ring of the form $S[[X_1, \dots, X_n]]/(f)$ where S is a complete non-Gorenstein local ring and f is a monomial, then there are infinitely many isomorphism classes of indecomposable totally reflexive R-modules.

In the next section, we will prove Theorem 1.3 by using a theorem of Huneke and Leuschke [9] and Theorem 1.2. In the last section, we will give several applications of Theorem 1.3.

2. Proof of the theorem

In this note, $(R, \mathfrak{m}, k)$ is always a commutative noetherian local ring, and all R-modules are finitely generated. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. We begin with recalling the definition of a resolving subcategory.

Definition 2.1. A full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if the following hold.
(1) $\mathcal{X}$ contains R.
(2) $\mathcal{X}$ is closed under direct summands: if $M \in \mathcal{X}$ and N is a direct summand of M, then $N \in \mathcal{X}$.
(3) $\mathcal{X}$ is closed under extensions: if there is an exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$ with $L, N \in \mathcal{X}$, then $M \in \mathcal{X}$.
(4) $\mathcal{X}$ is closed under kernels of epimorphisms: if there is an exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$ with $M, N \in \mathcal{X}$, then $L \in \mathcal{X}$.

In this definition, the condition (3) especially says that $\mathcal{X}$ is closed under finite direct sums: if $M, N \in \mathcal{X}$, then $M \oplus N \in \mathcal{X}$. Hence from (1) it follows that $\mathcal{X}$ contains all free R-modules. Therefore, by (4), $\mathcal{X}$ is closed under syzygies: the (first) syzygy of any R-module in $\mathcal{X}$ is also in $\mathcal{X}$.

For an R-module M, we denote by $e(M)$ ($\nu(M)$, respectively) the multiplicity (the minimal number of generators, respectively) of M, namely,

$$\begin{cases} e(M) = \lim_{n\to\infty} \frac{d!}{n^d} \ell_R(M/\mathfrak{m}^n M), \\ \nu(M) = \dim_k(M \otimes_R k), \end{cases}$$

where $d = \dim M$ and $\ell_R(N)$ denotes the length of an R-module N. Huneke and Leuschke essentially proved the following theorem in [9, Theorems 1,3]. (They actually proved the theorem in the case where $\mathcal{X}$ is the category of maximal Cohen-Macaulay R-modules.)

Theorem 2.2 (Huneke-Leuschke). *Let $\mathcal{X}$ be a full subcategory of* $\operatorname{mod} R$ *which is closed under extensions.*

1. *Let $M, N \in \mathcal{X}$. Assume that there are only finitely many isomorphism classes of R-modules $X \in \mathcal{X}$ with $e(X) = e(M) + e(N)$, and denote by h the number of such isomorphism classes. Then $\mathfrak{m}^h \operatorname{Ext}^1_R(M, N) = 0$.*
2. *Suppose that $\mathcal{X}$ is resolving. Let $M \in \mathcal{X}$. Assume that there are only finitely many isomorphism classes of indecomposable R-modules $X \in \mathcal{X}$ with $e(X) \le \nu(M) \cdot e(R)$. Then $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free for any $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$.*

As a special case of the second assertion of this theorem, we obtain the following.

Corollary 2.3. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$. *Suppose that there are only finitely many isomorphism classes of indecomposable R-modules in $\mathcal{X}$. Then $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free for any $M \in \mathcal{X}$ and $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$.*

Next, we recall the definition of a totally reflexive module. Let $(-)^*$ be the R-dual functor $\operatorname{Hom}_R(-, R)$.

Definition 2.4. We say that an R-module M is *totally reflexive* (or M has *G-dimension zero*) if the natural homomorphism $M \to M^{**}$ is an isomorphism and $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ for any $i > 0$.

We denote by $\mathcal{G}$ the full subcategory of $\operatorname{mod} R$ consisting of all totally reflexive R-modules. Here, we state the properties of $\mathcal{G}$ which will be used later.

Lemma 2.5. (1) *$\mathcal{G}$ is a resolving subcategory of* $\operatorname{mod} R$.

(2) *$\mathcal{G}$ is closed under R-dual, syzygies and finite direct sums.*

(3) *For any $M \in \mathcal{G}$, one has* $\operatorname{depth} M = \operatorname{depth} R$.

Proof. (1) We refer to [3, Lemma 2.3], for example.

(2) It is easy to see from definition that if $M \in \mathcal{G}$ then $M^* \in \mathcal{G}$. The remaining assertions follow from the arguments following Definition 2.1.

(3) See [2, Proposition (4.12)] or [4, Theorem (1.4.8)]. □

Proposition 2.6. *Suppose that there is a nonfree cyclic totally reflexive R-module M such that $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free for any $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$. Then* $\operatorname{depth} R \le 1$.

Proof. Suppose that $\operatorname{depth} R \ge 2$. We want to derive a contradiction. We may assume $M = R/I$, where I is an ideal of R with $0 \ne I \subseteq \mathfrak{m}$. Setting $J = (0 : I)$, we have $J \ne R$, hence $I + J \subseteq \mathfrak{m}$. Dualizing the natural exact sequence $0 \to I \xrightarrow{\theta} R \to R/I \to 0$ and using that R/I is assumed to be totally reflexive, gives an exact sequence

$$0 \to J \to R \xrightarrow{\kappa} I^* \to \operatorname{Ext}^1(R/I, R) = 0,$$

where $\kappa(1) = \theta$. Thus we get an isomorphism $\lambda : R/J \to I^*$, where $\lambda(\overline{1}) = \theta$. Lemma 2.5(2) says that the R-modules I and I^* are totally reflexive. Hence so is R/J, and there are isomorphisms

$$I \to I^{**} \xrightarrow{\lambda^*} (R/J)^* \to (0 : J).$$

It is easy to check that the composite of these isomorphisms is an identity map; we obtain $I = (0 : J)$.

Fix $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$. Since $(R/I)_\mathfrak{p}$ is $R_\mathfrak{p}$-free, one has either $IR_\mathfrak{p} = 0$ or $IR_\mathfrak{p} = R_\mathfrak{p}$. If $IR_\mathfrak{p} = 0$, then $\mathfrak{p}$ does not belong to $\operatorname{Supp} I$. Noting that $\operatorname{Supp} I = V((0 : I)) = V(J)$, we see that J is not contained in $\mathfrak{p}$. If $IR_\mathfrak{p} = R_\mathfrak{p}$, then I is not contained in $\mathfrak{p}$. This means that the ideal $I + J$ is $\mathfrak{m}$-primary. There is an exact sequence

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I + J) \to 0.$$

Since the R-module $R/(I + J)$ has finite length, we have $\operatorname{depth}(R/(I + J)) = 0$. According to Lemma 2.5 parts (2) and (3), we get $\operatorname{depth}(R/I \oplus R/J) = \operatorname{depth} R \geq 2 > 0$. Hence the depth lemma (cf. [10, Proposition 4.3.1]) yields $\operatorname{depth}(R/(I \cap J)) = 1$.

Let $x \in I \cap J$. Then, since $I = (0 : J)$ and $J = (0 : I)$, one has $xJ = xI = 0$, which implies that $I + J \subseteq (0 : x)$. As $I + J$ is an $\mathfrak{m}$-primary ideal, so is $(0 : x)$. Hence $\mathfrak{m}^r x = 0$ for some $r > 0$. It follows that $I \cap J$ is an R-module of finite length. Noting that $\operatorname{depth} R \geq 2 > 0$, one must have $I \cap J = 0$. Thus $2 \leq \operatorname{depth} R = \operatorname{depth}(R/(I \cap J)) = 1$. This contradiction proves the proposition. □

Now we can prove our main theorem.

Proof of Theorem 1.3. Suppose that $\mathcal{G}$ has only finitely many isomorphism classes of indecomposable R-modules. Then Theorem 1.2 implies that $\operatorname{depth} R \geq 3$. But $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free for any $M \in \mathcal{G}$ and $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$ by Lemma 2.5(1) and Corollary 2.3, hence $\operatorname{depth} R \leq 1$ by Proposition 2.6. This is a contradiction, which completes the proof of the theorem. □

3. Applications

In this section, using Theorem 1.3, we give several results on the number of indecomposable totally reflexive modules.

Corollary 3.1. *Let $(R, \mathfrak{m})$ be a henselian non-Gorenstein local ring. If there exist $x, y \in \mathfrak{m}$ such that $(0 : x) = (y)$ and $(0 : y) = (x)$, then there exist infinitely many nonisomorphic indecomposable totally reflexive R-modules.*

Proof. Noting that there is an exact sequence $\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} \cdots$, we can easily check that the R-module $R/(x)$ is nonfree totally reflexive. Hence the assertion follows from Theorem 1.3. □

Corollary 3.2. *Let R be a complete non-Gorenstein local ring. Then $S = R[[X_1, X_2, \ldots, X_n]]/(X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n})$ admits infinitely many nonisomorphic indecomposable totally reflexive modules.*

Proof. Note that S is faithfully flat over R. Hence S is also a complete non-Gorenstein local ring. To show the corollary, we may assume $a_1 > 0$. Then it is easily seen that $(0 :_S X_1) = X_1^{a_1-1} X_2^{a_2} \cdots X_n^{a_n} S$ and $(0 :_S X_1^{a_1-1} X_2^{a_2} \cdots X_n^{a_n}) = X_1 S$. Thus we can apply Corollary 3.1. □

We denote by $\mathfrak{D}(R)$ the derived category of $\operatorname{mod} R$. Recall that an R-module C is called *semidualizing* if the natural morphism $R \to \mathbf{R}\mathrm{Hom}_R(C,C)$ is an isomorphism in $\mathfrak{D}(R)$, equivalently, the natural homomorphism $R \to \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}^i_R(C,C) = 0$ for any $i > 0$. In the following, we consider the idealization $S = R \ltimes C$ of a semidualizing module C over R. There are two natural homomorphisms $\phi : R \to S$ and $\psi : S \to R$, which are given by $\phi(a) = (a,0)$ and $\psi(a,x) = a$. Through the homomorphism ϕ (ψ, respectively), one can regard an S-module (R-module, respectively) as an R-module (S-module, respectively). Note that through the composite of these homomorphisms the R-module structure is preserved since $\psi\phi$ is the identity map of R, but the S-module structure is not preserved in general.

Lemma 3.3. [8, Lemma 3.2] *Let C be a semidualizing R-module, and set $S = R \ltimes C$. Then there is a natural isomorphism $\mathbf{R}\mathrm{Hom}_R(-,C) \cong \mathbf{R}\mathrm{Hom}_S(-,S)$ of functors on $\mathfrak{D}(R)$.*

Using this lemma, we can get the following result. It says that a non-Gorenstein ring which is the idealization of a semidualizing module over a henselian local ring has infinitely many nonisomorphic totally reflexive modules.

Corollary 3.4. *Let R be a henselian local ring, C a semidualizing R-module, and $S = R \ltimes C$ the idealization. Suppose that there are only finitely many nonisomorphic indecomposable totally reflexive S-modules. Then S is Gorenstein. Hence R is Cohen-Macaulay, and C is a canonical module of R.*

Proof. The last assertion follows from [5, Theorem 5.6]. (One can also prove it by using the isomorphism $\mathbf{R}\mathrm{Hom}_R(k,C) \cong \mathbf{R}\mathrm{Hom}_S(k,S)$ induced by Lemma 3.3.) Lemma 3.3 gives isomorphisms $\mathbf{R}\mathrm{Hom}_S(R,S) \cong C$ and $\mathbf{R}\mathrm{Hom}_S(C,S) \cong \mathbf{R}\mathrm{Hom}_R(C,C) \cong R$. We easily see from these isomorphisms that R is a totally reflexive S-module. Note that R is a nonfree cyclic S-module, and S is henselian since S is module-finite over R. Therefore, Theorem 1.3 implies that S is Gorenstein. $\square$

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